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IMPULSIVE MOTION OF ELASTO-PLASTIC BEAMS*

by

H.H. Bleich** and M.G. Salvadori**

SYNOPSIS

A general analysis of the vibrations of elasto-plastic structures is given and applied to free-free beams moving under dynamic forces. The motion during the successive elastic and elasto-plastic stages is expressed in terms of the normal modes of the structure and the maximum plastic angle is evaluated. The case of impulsive loads (initial velocities), which is of practical interest, is specifically considered, and by means of simplifying assumptions, upper and, often, lower bounds of the plastic angle are determined.

I. GENERAL THEORY.

Consider an ideally elasto-plastic body capable of vibrating elastically in a single infinity of normalized, orthogonal, vectorial modes $\phi_i(P)$ $(i = 0, 1, 2, \ldots)$. When the body is subjected to a system of external time varying forces, symbolically indicated by $F(t)$, the vectorial elastic displacement $u_e(P)$ of one of its points P may be expanded into a series of modes

$$u_e(P) = \sum_{i=0}^{\infty} q_i(t) \quad \phi_i(P), \tag{1}$$

where the generalized coordinates q_i, in the absence of damping, satisfy the Lagrangian equations

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) + \frac{\partial V}{\partial q_i} = Q_i \quad (i = 0, 1, 2, \ldots). \tag{2}$$

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In Eq. (2), T is the kinetic energy and V is the potential energy of the elastic body, and the Q_i are the generalized forces corresponding to the system of external forces F.

Let us now assume that at a time $t = t_o$ a plastic state of stress σ_o is reached (Fig. 1) in a region of area A_o about the point P_o, while at all other points the stresses remain elastic. Then for $t > t_o$ we may consider the constant force $F_o = \sigma_o A_o$ as an external force applied to the same body, in which the continuity is destroyed on the area A_o as far as the strain ε corresponding to σ is concerned. Indicating the normalized modes of this new, relaxed body by $\psi_i(P)$[1], its generalized coordinates by $p_i(t)$ and the generalized force corresponding to $F(t) + F_o$ by P_i, the vectorial elasto-plastic displacement $u_p(P)$ may be expanded in a series

$$u_p(P) = \sum_{i=0}^{\infty} p_i(t)\ \psi_i(P), \tag{3}$$

where the p_i satisfy the Lagrangian equations

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) + \frac{\partial V}{\partial q_i} = P_i\ . \tag{4}$$

The continuity of the displacement u and of the velocity $\dot{u}$ at $t = t_o$ require that:

$$u_e(P)]_{t=t_o} = u_p(P)]_{t=t_o}\ ; \quad \dot{u}_e(P)]_{t=t_o} = \dot{u}_p(P)]_{t=t_o}\ , \tag{5}$$

i.e., that:

$$\begin{aligned} \sum_{i=0}^{\infty} q_i(t_o)\ \phi_i(P) &= \sum_{i=0}^{\infty} p_i(t_o)\ \psi_i(P) \\ \sum_{i=0}^{\infty} \dot{q}_i(t_o)\ \phi_i(P) &= \sum_{i=0}^{\infty} \dot{p}_i(t_o)\ \psi_i(P)\ . \end{aligned} \tag{6}$$

(1) The modes ψ_i of the relaxed body may be conveniently obtained, in many cases, by the method of "complementary systems" (see H.H. Bleich, "Frequency Analysis of Beam and Girder Floors", Transactions, ASCE, Vol. 115, 1950, p. 1023).

Multiplying Eqs. (6) by ψ_k, integrating over the entire mass of the body and remembering the orthogonality conditions of the modes ψ_i:

$$\int \psi_i \, \psi_k \, dm = \begin{cases} 0 & i \neq k \\ 1 & i = k \end{cases}, \tag{7}$$

the initial conditions for the p_i reduce to:

$$\begin{aligned} p_k(t_o) &= \sum_{i=o}^{\infty} b_{ki} \, q_i(t_o) \\ \dot{p}_k(t_o) &= \sum_{i=o}^{\infty} b_{ki} \, \dot{q}_i(t_o) , \end{aligned} \tag{8}$$

where:

$$b_{ki} = \int_m \phi_i \, \psi_k \, dm . \tag{9}$$

Similar relaxations will give the equations of motion when other areas of the body become successively plastic, provided the plastic areas do not change shape or location.

The elasto-plastic motion defined by Eq. (3) will continue until the stresses in the plastic area A_o go back into the elastic range. If this reversal takes place at a time, say t_1, the displacement $\bar{u}_e(P)$ for $t > t_1$ is given again by Eq. (1) with new generalized coordinates $\bar{q}_i$, satisfying Eqs. (2) and the initial conditions of continuity

$$u_p(P)]_{t=t_1} = \bar{u}_e(P)]_{t=t_1}; \quad \dot{u}_p(P)]_{t=t_1} = \dot{\bar{u}}_e(P)]_{t=t_1} \tag{10}$$

or

$$\begin{aligned} \sum_{i=o}^{\infty} p_i(t_1) \, \psi_i(P) &= \sum_{i=o}^{\infty} \bar{q}_i(t_1) \, \phi_i(P) \\ \sum_{i=o}^{\infty} \dot{p}_i(t_1) \, \psi_i(P) &= \sum_{i=o}^{\infty} \dot{\bar{q}}_i(t_1) \, \phi_i(P) . \end{aligned}$$

Multiplying these equations by $\phi_k(P)$ and integrating over the whole mass of the body, the initial conditions for the $\bar{q}_i$ become, because of the orthogonality of the modes ϕ_i :

$$\bar{q}_k(t_1) = \sum_{i=0}^{\infty} \bar{b}_{ki}\, p_i(t_1)$$
$$\dot{\bar{q}}_k(t_1) = \sum_{i=0}^{\infty} \bar{b}_{ki}\, \dot{p}_i(t_1) \,, \tag{11}$$

where

$$\bar{b}_{ki} = b_{ik} \,. \tag{12}$$

A sequence of changes from the elastic to the plastic range and vice versa allows the description of the elasto-plastic forced vibrations of the body at any time t.

Under particular circumstances, one or more of the plastic regions A may either change shape or location or both before a reversal of stress or before additional plastic regions appear. In these cases (which are not considered in the present paper), new and more complicated equations of motion must be set up and their integration may have to be performed by approximate, step-by-step methods.

2. THE ELASTO-PLASTIC VIBRATIONS OF FREE-FREE BEAMS.

As a typical example of application of the general theory of Sec. 1, consider the forced vibrations of a free-free beam of length $2l$, variable flexural rigidity EI and variable mass per unit length m (Fig. 2).

The free vibrations of such a beam are governed by the differential system:

$$(EIy'') + m\ddot{y} = 0 \tag{13}$$

$$y''(\pm l, t) = y'''(\pm l, t) = 0 \tag{14}$$

$$y(x, 0) = y_o(x) \,; \quad \dot{y}(x, 0) = \dot{y}_o(x) \,, \tag{15}$$

where primes denote differentiation with respect to the variable x, measured along the centroidal axis of the beam with origin at its middle, and dots denote differentiation with respect to time.

The substitution of

$$y(x, t) = \phi(x)q(t) \tag{16}$$

in Eqs. (13), (14), and (15) shows that the beam can vibrate in an infinite number of modes ϕ_i, satisfying the equations

$$(EI\ \phi_i'')'' - \omega_i^2\, m\ \phi_i = 0 \quad (i = 0, 1, 2, \ldots) \tag{17}$$

and the boundary conditions:

$$\phi_i''(\pm l) = \phi_i'''(\pm l) = 0 . \tag{18}$$

The modes ϕ_i are orthogonal and may be normalized so that

$$\int_{-l}^{l} \phi_i(x)\ \phi_j(x)\, dm = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases} \tag{19}$$

It is convenient to indicate by the subscript i the number of nodes in the i-th mode: thus, $i = 0$ represents the rigid translation of the beam, $i = 1$ its rigid rotation, $i = 2$ its first elastic mode, etc.

Indicating the general integral of Eq. (17) by

$$\phi_i = c_1 z_1 + c_2 z_2 + c_3 z_3 + c_4 z_4 \tag{20}$$

and substituting this value of ϕ_i in Eqs. (18), we obtain a homogeneous system of 4 linear algebraic equations in the constants c_1 to c_4, whose determinant set equal to zero determines the frequency ω_i of the i-th mode.

The modes ϕ_i can be evaluated rigorously from the differential system (17), (18), or, if this is impossible or inconvenient, they can be obtained by approximate numerical or experimental methods.

The forced elastic deflections $y_e(x, t)$ of a free-free beam due to an external, time varying pressure $p(x, t)$ per unit length may be expanded in an infinite series of modes

$$y_e(x, t) = \sum_{i=0}^{\infty} \phi_i(x)\, q_i(t) . \tag{21}$$

Substitution of Eq. (21) in the Lagrangian Eqs. (2) and in the initial conditions (15) shows that the generalized coordinates q_i satisfy the equations:

$$\ddot{q}_i + \omega_i^2 q_i = Q_i \tag{22}$$

and the initial conditions

$$\sum_{i=0}^{\infty} \phi_i(x)\, q_i(0) = y_o(x)$$
$$\sum_{i=0}^{\infty} \phi_i(x)\, \dot{q}_i(0) = \dot{y}_o(x) \; . \tag{23}$$

The generalized forces Q_i corresponding to the load $p(x, t)$ are given by the integrals

$$Q_i(t) = \int_{-\ell}^{\ell} p(x, t)\; \phi_i(x)\, dx \; . \tag{24}$$

Multiplying Eqs. (23) by $\phi_j dm$ and integrating over the beam length, the initial conditions for the q_i become, by Eqs. (19):

$$q_i(0) = \int_{-\ell}^{\ell} y_o(x)\; \phi_i(x)\, dm$$
$$\dot{q}_i(0) = \int_{-\ell}^{\ell} \dot{y}_o(x)\; \phi_i(x)\, dm \; . \tag{25}$$

At any time t, during the elastic stage of the vibrations, the bending moment M at a section x is obtained by double differentiation of the deflection:

$$M(x, t) = -EIy''_e = -EI \sum_{i=0}^{\infty} \phi''_i(x)\, q_i(t) \; . \tag{26}$$

Under the assumption of an ideal elasto-plastic material the elastic stage of the vibrations ends at the smallest value of the time t, say t_o, at which the moment M reaches the plastic capacity value M_o anywhere in the beam. If the moment M reaches the capacity value first at a section $x = \bar{x}$, the elastic stage of the vibrations ends when

$$\left| M(\bar{x}, t_o) \right| = \left| -EI \sum_{i=0}^{\infty} \phi''_i(\bar{x})\, q_i(t_o) \right| = M_o(\bar{x}) \; , \tag{27}$$

where $M_o(\bar{x})$ is the capacity moment of the beam at the section $x = \bar{x}$.

If, at least for a time, the capacity moment is not reached at any other section, the moment $M(\bar{x}, t)$ will remain constant

and equal to $M_0(\bar{x})$. The elasto-plastic vibrations of the beam are then identical with the vibrations of another beam equal to the beam considered, but with a hinge at $x = \bar{x}$, and acted upon by the same load $p(x, t)$ and by two external, equal and opposite moments $M_0(\bar{x})$ at the hinge.

Indicating by $\psi_i(x)$ the orthonormal modes of the hinged beam, the elasto-plastic vibrations of the original beam may be expanded into a series

$$y_p(x, t) = \sum_{i=0}^{\infty} \psi_i(x)\, p_i(t) , \tag{28}$$

where the new generalized coordinates p_i satisfy the differential equations:

$$\ddot{p}_i + \Omega_i^2\, p_i = P_i , \tag{29}$$

Ω_i being the frequency of the i-th mode of the hinged beam, and P_i the generalized forces due to the load p and the moments $M_0(\bar{x})$. The generalized force P_i is thus given by

$$P_i(t) = \int_{-l}^{l} p(x,t)\, \psi_i(x)dx + M_0(\bar{x})[\, \psi'_i(\bar{x} + 0) - \psi'_i(\bar{x} - 0)] . \tag{30}$$

The conditions of continuity of displacements and velocities at $t = t_0$ require that

$$y_e(x,t_0) = y_p(x,t_0) \qquad \dot{y}_e(x,t_0) = \dot{y}_p(x,t_0) . \tag{31}$$

Substitution of Eqs. (21) and (28) in these conditions, multiplication by $\psi_k(x)\, dm$ and integration over the whole beam, reduce these equations to the form:

$$\begin{aligned} p_k(t_0) &= \sum_{i=0}^{\infty} b_{ki}\, q_i(t_0) \\ \dot{p}_k(t_0) &= \sum_{i=0}^{\infty} b_{ki}\, \dot{q}_i(t_0) \end{aligned} \tag{32}$$

where the b_{ki} are given by Eqs. (9).

When a beam has mass symmetrically distributed about its middle point and the pressure $p(x, t)$ is also symmetrical and such that the first hinge will develop at the middle point, the modes ψ_i of the hinged beam are obtained by noticing that there is no force transmitted through the hinge and that each portion of the beam vibrates as a free-free beam of length l. The modes ψ_i are the modes of the two half beams normalized over the entire beam.

While the moment at $x = \bar{x}$ remains constant, a plastic angle $\theta(t)$ develops at this section; its value equals the discontinuity of the slope of the elasto-plastic deflection at $x = \bar{x}$:

$$\theta(t) = y'_p(\bar{x} + 0, t) - y'_p(\bar{x} - 0, t) = \sum_{i=0}^{\infty} [\psi'_i(\bar{x} + 0) - \psi'_i(\bar{x} - 0)] p_i(t) . \tag{33}$$

When at time, say t_1, this angle reaches a maximum value, the relative motion of the two portions of the beam is reversed and the moment at $x = \bar{x}$ becomes elastic again. Under the assumption of small plastic deflections and of elastic stresses at all sections of the beam except at $x = \bar{x}$, the beam deflections are given again by the series (21) with new generalized coordinates $\bar{q}_i$, satisfying Eqs. (22) and conditions of continuity at $t = t_1$. Since the actual computations of a specific problem cannot be carried out with more than the first few terms of the series expansions (21) and (28) (3 to 5 terms are usually sufficient), conditions of continuity on the displacements y and on the velocities $\dot{y}$ at $t = t_1$ do not necessarily make the accelerations continuous at that time and hence may give a numerical discrepancy in the values of the moment $M(\bar{x}, t_1)$ computed by inertia forces in the elasto-plastic range and according to Eq. (26) in the elastic range. It is therefore often advisable to obtain the initial values of the $\bar{q}$ by

imposing conditions of continuity on the velocities and the accelerations, in order to obtain continuous stresses at $t = t_1$. These conditions require that:

$$\dot{y}_p(x, t_1) = \dot{\bar{y}}_e(x, t_1); \quad \ddot{y}_p(x, t_1) = \ddot{\bar{y}}_e(x, t_1) \tag{34}$$

and, by means of Eqs. (21), (28) and (19) , reduce to:

$$\begin{aligned} \dot{\bar{q}}_k(t_1) &= \sum_{i=0}^{\infty} \bar{b}_{ki}\, \dot{p}_i(t_1) \\ \ddot{\bar{q}}_k(t_1) &= \sum_{i=0}^{\infty} \bar{b}_{ki}\, \ddot{p}_i(t_1) \ , \end{aligned} \tag{35}$$

where

$$\bar{b}_{ki} = \int \phi_k \ \psi_i \, dm = b_{ik} \ . \tag{36}$$

Under the assumptions made, the stresses in the beam may be evaluated at any time t in terms of successive elastic and elasto-plastic deformations of the type given by Eqs. (21) and (28). In some cases the successive development of plastic hinges at more than one section must be considered, but the new hinges can be treated as shown above and introduce no essential difficulty in the solution. In other cases, one or more of the successive hinges may shift their location continuously so that a whole portion of the beam becomes plastic[1]; these wandering hinges introduce essential difficulties in the integration of the problem, which then can e.g. be solved step-by-step, using numerical integration methods. Alternatively, such cases can be treated approximately by assuming that the beam has one or more "weak" points in the plastic region; all plastic deformations then occur at these "weak" points, and moments slighly in excess of M_o outside the "weak" points are tolerated.

(1) See, E.H. Lee and P.S. Symonds. "Large Plastic Deformations of Beams under Transverse Impact", Trans. ASME, Vol. 74. 1952. p. 308; M.G. Salvadori and F. DiMaggio, "On the Development of Plastic Hinges in Rigid-Plastic Beams , Quart. App. Math., July 1953, p. 223 ff.

3. IMPULSIVE MOTION OF FREE-FREE BEAMS WITH CONSTANT FLEXURAL RIGIDITY.

3.1. The elastic stage.

The impulsive motion of a beam due to forces of high intensity and short duration is equivalent, in the limit, to the motion due to a sudden initial velocity. As an illustration of the theory presented in the previous sections we shall derive here the formulas governing the elasto-plastic motion of a free-free beam of length $2l$ and constant flexural rigidity EI, whose initial velocity $v_0(x)$ is given by the symmetrical distribution (Fig. 3):

$$\dot{y}_0(x) = \frac{v_0}{2}\left(1 + \cos\frac{\pi}{l}x\right). \tag{37}$$

The elastic orthonormal modes of such a beam are given by[1]

$$\phi_0 = \frac{1}{\sqrt{2ml}}\,; \quad \phi_1 = \frac{\sqrt{3}}{\sqrt{2ml}}\,\frac{x}{l}$$

$$\phi_i \begin{cases} = \dfrac{1}{\sqrt{2ml}}\left(\dfrac{\cos k_i x}{\cos k_i l} + \dfrac{\cosh k_i x}{\cosh k_i l}\right) & i = 2r \\ \qquad\qquad (r = 1, 2, ..) \\ = \dfrac{1}{\sqrt{2ml}}\left(\dfrac{\sin k_i x}{\sin k_i l} + \dfrac{\sinh k_i x}{\sinh k_i l}\right) & i = 2r + 1 \end{cases} \tag{38}$$

and the corresponding frequencies by:

TABLE I

i	0	1	2	3	4	5	6
$2k_i l$	0	0	4.730	7.853	10.996	14.137	17.279

with

$$(k_i l)^4 = \frac{m l^4}{EI}\,\omega_i^2 \tag{39}$$

where $k_i l$ are the roots of the transcendental equation

(1) S. Timoshenko, "Vibration Problems in Engineering", D. Van Nostrand Co., Inc., New York, second edition, p. 342 ff.

$$\cos 2k\ell \cosh 2k\ell = 1 .^{(1)} \qquad (40)$$

The generalized coordinates q_i, in the absence of the external pressure $p(x, t)$, satisfy the homogeneous equations

$$\ddot{q}_i + \omega_i^2 q_i = 0 \quad (i = 2r) \qquad (41)$$

and, since $y_0(x)$ is assumed equal to zero, the initial conditions (26):

$$q_i(0) = \int_{-\ell}^{\ell} y_0(x) \; \phi_i(x) \, dm = 0 \qquad (i = 2r) \qquad (42)$$

$$\dot{q}_i(0) = \int_{-\ell}^{\ell} \dot{y}_0(x) \; \phi_i(x) \, dm .$$

Hence, the q_i are given by:

$$q_0(t) = \dot{q}_0(0) \, t = \left(\frac{v_0}{2} \sqrt{2m\ell}\right) t \qquad (43a)$$

$$q_i(t) = q_i(0) \cos \omega_i t + (\dot{q}_i(0)/\omega_i) \sin \omega_i t$$

$$= (\dot{q}_i(0)/\omega_i) \sin \omega_i t \qquad (i = 2r) \qquad (43b)$$

b. The first elasto-plastic stage.

The symmetrical initial velocity (37) excites only the even modes of the elastic beam and creates a bending moment diagram $M(x, t)$ with a maximum at $x = 0$, at least in the neighborhood of $t = 0$. Hence the section reaching plasticity first is the section $\bar{x} = 0$ and the "initial plasticity time" t_0 at which the first plastic hinge occurs is determined by the equation

$$\left| \sum_{i=0}^{\infty} \phi''_{2r}(0) \, q_{2r}(t_0) \right| = M_0(0)/EI . \qquad (44)$$

The modes ψ of the hinged beam are obtained by noticing that each half-beam vibrates like a free-free beam and that their common end must have the same deflection. Hence each half-beam has modes obtainable from those of Eqs. (38) by

(1) The roots of Eq. (40) approach asymptotically the values $(i - 1/2)\,\pi$ as i approaches infinity.

changing l into $l/2$ and by normalizing the new modes over the entire length $2l$ of the beam. Both the symmetrical and the antisymmetrical modes of the half-beams must be considered, and the corresponding frequencies are obtained from Table I by changing kl into $kl/2$. Thus, the modes, referred to the middle point of the half-beams, become:

$$\psi_0 = \frac{1}{\sqrt{2ml}} \quad ; \quad \psi_2 = \frac{\sqrt{3}}{\sqrt{2ml}} \frac{2x'}{l}$$

$$\psi_{2i} = \begin{cases} \dfrac{1}{\sqrt{2ml}} \left(\dfrac{\cos 2k_i x'}{\cos k_j l} + \dfrac{\cosh 2k_i x'}{\cosh k_i l} \right) \\ \qquad\qquad (-\tfrac{l}{2} \leq x' \leq \tfrac{l}{2}), (i = 2,3,4) \\ \dfrac{1}{\sqrt{2ml}} \left(\dfrac{\sin 2k_i x'}{\sin k_i l} + \dfrac{\sinh 2k_i x'}{\sinh k_i l} \right) , \end{cases} \tag{45}$$

where the k_j are taken from Table I and $x' = x - l/2$.

The elasto-plastic generalized coordinates p_i satisfy the non-homogeneous equations

$$\ddot{p}_{2i} + \Omega_{2i}^2 p_{2i} = P_{2i} \ , \tag{46}$$

with

$$\Omega_{2i}^2 = 2^4 \cdot \omega_i^2 \tag{47}$$

and the initial conditions (32), in which the constants b_{ik} are computed by substituting the values of ϕ_i and ψ_k of Eqs. (38) and (45) in Eq. (9). The generalized forces P_i due to the moments $M_0(0)$ are given by

$$P_{2i} = M_0(0) \, \theta_{2i} \ , \tag{48}$$

where the angle θ_{2i} equals the slope discontinuity of the i-th mode:

$$\theta_{2i} = \psi'_{2i}(0^+) - \psi'_{2i}(0^-) = 2 \, \psi'_{2i}(0) \ . \tag{49}$$

Hence the generalized coordinates p_i take the form

$$p_{2i}(t) = \begin{cases} \dfrac{P_{2i}(t - t_0)^2}{2} + \dot{p}_{2i}(t_0)(t - t_0) + p_{2i}(t_0) \\ \qquad (i = 0, 1) \qquad (50) \\ A_{2i} \cos \Omega_{2i} t + B_{2i} \sin \Omega_{2i} t + \dfrac{P_{2i}}{\Omega_{2i}^2} \\ \qquad (i = 2, 3, ..) \qquad (51) \end{cases}$$

where

$$A_{2i} = \left(p_{2i}(t_0) - \frac{P_{2i}}{\Omega_{2i}^2}\right) \cos \Omega_{2i} t_0 - \frac{\dot{p}_{2i}(t_0)}{\Omega_{2i}} \sin \Omega_{2i} t_0$$

$$B_{2i} = \left(p_{2i}(t_0) - \frac{P_{2i}}{\Omega_{2i}^2}\right) \sin \Omega_{2i} t_0 + \frac{\dot{p}_{2i}(t_0)}{\Omega_{2i}} \cos \Omega_{2i} t_0$$

The elasto-plastic displacements of the beam are computed by substitution of the ψ_{2i}, given by Eqs. (45), and p_{2i}, given by Eqs. (50), (51) in Eq. (28) and the plastic angle at $\bar{x} = 0$ is computed from Eq. (33), which becomes in this case

$$\theta(t) = 2 \sum_{i=0}^{\infty} \psi'_{2i}(0)\, p_{2i}(t) \,. \qquad (52)$$

The maximum absolute value $|\theta|_{max}$ of the plastic angle θ is of practical importance both because it determines the amount of energy, $M_0(\bar{x})\theta_{max}$, spent in plastic flow and because the time, say t_1, of its occurrence establishes the transition from the elasto-plastic to the elastic stages. The time t_1 is determined by setting equal to zero the derivative of $\theta(t)$, as given by Eq. (52), that is, by the equation:

$$\sum_{i=0}^{\infty} \psi'_{2i}(0)\, \dot{p}_{2i}(t_1) = 0 \,. \qquad (53)$$

An upper bound for $|\theta|_{max}$ may be easily obtained by assuming that the available energy of impact which is not transformed

into translational energy goes entirely into plastic flow. By allowing that part of the energy which would actually be spent in elastic vibrations to be available for plastic work, the value of $|\Theta|_{max}$ obtained is larger than the true value. The upper bound for $|\Theta|_{max}$ is hence computed from Eq. (52) by considering only the rigid rotation mode ψ_2. The plastic hinge occurs at $t_0 = 0$, at which time the total energy in the system is the kinetic energy $m \int_o^l [v(x)]^2 dx$. Considering the beam as having a plastic hinge at the center but otherwise as rigid, the amount of energy available for deformation will be the kinetic energy at t = 0 reduced by the kinetic energy of the translation, $\frac{1}{2} [\dot{p}_0(0)]^2$. Placing all the remaining energy as kinetic energy $\frac{1}{2} [\dot{p}_2(0)]^2$ in the second mode must therefore give an upper bound for the angle $\Theta(t)$. Thus with

$$p_2(0) = \int_{-l}^{l} y_0(x) \quad \psi_2(x) \, dm = 0 \tag{54a}$$

$$\frac{1}{2} [\dot{p}_2(0)]^2 = m \int_o^l [v(x)]^2 dx - \frac{1}{2} [\dot{p}_0(0)]^2 \tag{54b}$$

we obtain an upper bound $\Theta_u(t)$ for $\Theta(t)$:

$$\Theta_u(t) = 2 \psi'_2(0) \left[\frac{P_2 t^2}{2} + \dot{p}_2(0) t \right] . \tag{55}$$

The maximum absolute value of the upper bound $\Theta_u(t)$ occurs at a time:

$$t_1 = - \dot{p}_2(0)/P_2 \tag{56}$$

and equals:

$$|\Theta_u|_{max} = \frac{\psi'_2(0)}{P_2} [\dot{p}_2(0)]^2 > |\Theta_{max}| \tag{57}$$

where $\dot{p}_2(0)$ is given by Eq. (54b). P_2 is found from Eqs. (48) and (49), $P_2 = 2 \psi'_2(0) M_0(0)$.

The value $|\Theta_u|_{max}$ is a good approximation to the true value $|\Theta|_{max}$ whenever the initial velocity $v_0(x)$ is so large

that the maximum bending moment which would develop at O if the beam never became plastic is several times larger than the capacity moment. Thus even if the impact time is very short, the maximum plastic angle may be well approximated by the rigid-plastic approach, although the time history of the angle is incorrect due to the assumption of a plasticity time t_0 equal to zero.

Another approximation θ_l of θ, which gives a more correct time history and is often a lower bound, may be obtained by freezing the beam into a rigid plastic body at $t = t_0$ rather than at $t = 0$. In this case θ is obtained again from Eq. (52) by considering only the rigid rotation mode ψ_2 from $t = t_0$ on:

$$\theta_l(t) = 2\psi'_2(0)p_2(t) = 2\psi'_2(0)[\frac{P_2}{2}(t - t_0)^2 + \dot{p}_2(t_0)(t - t_0) + p_2(t_0)] \qquad (58)$$

The absolute value of θ_l becomes maximum at a time:

$$t_1 = t_0 - \dot{p}_2(t_0)/P_2 \qquad (59)$$

and equals:

$$|\theta_l|_{max} = (\psi'_2(0)/P_2)\,[(\dot{p}_2(t_0))^2 - 2P_2p_2(t_0)]\,, \qquad (60)$$

where $\dot{p}_2(t_0)$ and $p_2(t_0)$ are given by Eqs. (32).

It is seen that as $\dot{y}_0(x)$ increases and t_0 approaches zero, $|\theta_l|_{max}$ approaches $|\theta_u|_{max}$. Hence $|\theta_l|_{max}$ is also a good approximation of $|\theta|_{max}$ for values of $v_0(x)$ capable of creating elastic moments $M(0)$ several times larger than the capacity moment $M_0(0)$.

c. The successive elastic and elasto-plastic stages.

From $t = t_1$ on, the deflections become elastic again and are computed by means of the elastic modes (38) and of new generalized coordinates $\bar{q}_i$, satisfying the initial conditions

(35),

$$\bar{q}_i(t) = A_i \cos \omega_i t + B_i \sin \omega_i t , \qquad (i = 2r) \qquad (61)$$

where

$$A_i = - \frac{1}{\omega_i^2}[\ddot{\bar{q}}_i(t_1)\cos \omega_i t_1 + \omega_i \dot{\bar{q}}_i(t_1)\sin \omega_i t_1]$$

$$B_i = \frac{1}{\omega_i^2}[\omega_i \dot{\bar{q}}_i(t_1)\cos \omega_i t_1 - \ddot{\bar{q}}_i(t_1)\sin \omega_i t_1] \qquad (62)$$

$$\dot{\bar{q}}_k(t_1) = \sum_{i=0}^{\infty} \bar{b}_{ki} \dot{p}_i(t_1); \quad \ddot{\bar{q}}_k(t_1) = \sum_{i=0}^{\infty} \bar{b}_{ki} \ddot{p}_i(t_1) ;$$

the p_k are given by Eqs. (51) and the $\bar{b}_{ki} = b_{ik}$ by Eqs. (9).

In most cases of impulsive motion the consideration of further elasto-plastic stages is unnecessary, because of the energy spent in plastic flow, but successive elasto-plastic and elastic stages may be determined whenever needed by equations analogous to Eqs. (32) and (35).

4. A NUMERICAL EXAMPLE.

The theory presented in Sec. 3 is applied to a specific beam in order to illustrate its main features. The results of this application, in which the first 3 terms of the series were taken into account, are given graphically in this section.

The beam considered has the following characteristics in a consistent system of units:

Length: $2\ell = 320$
Flexural rigidity: $EI = 3.58 \times 10^8$
Mass per unit of length: $m = 0.800$
Capacity moment: $M_0(0) = 6.69 \times 10^4$
Fundamental period of the beam: $T = 1.36$

Two values of the initial velocity parameter v_0 were used:

$$v_0 = 11.1 ; \qquad v_0 = 67.4 .$$

The initial plasticity time t_0 evaluated from Eq. (44), was found to be

$$t_0 = 0.2 \quad \text{for} \quad v_0 = 11.1$$

$$t_0 = 0.016 \quad \text{for} \quad v_0 = 67.4 \ .$$

Fig. 4 gives the graph of the elastic moment at x = 0 for a full period of the fundamental mode of the beam, under the assumption of an infinitely large capacity moment throughout the beam. Fig. 5 gives the graph of the elasto-plastic moment at x = 0 for $v_0 = 11.1$ and for a full period of the fundamental elastic mode of the beam. It is seen from Figs. 4 and 5 that the maximum elastic moment M(0) would be about twice the capacity moment.

The moment becomes elastic again at time $t_1 = 0.375$ given by Eq. (53) and does not revert to its plastic value within the time considered.

Fig. 6 gives the plastic angle at x = 0, given by Eq. (52), together with the upper bound θ_u of Eq. (55) and the bound θ_l of Eq. (58), which in this case is a lower bound. For the low value $v_0 = 11.1$ of the impact velocity the lower bound for θ_{max} has an error of -56% and the upper bound has an error of +22%. The time at which the moment becomes elastic again ($t_1 = 0.375$) is approximated by $t_1 = 0.406$ by Eq. (59), derived from the lower bound, and by $t_1 = 0.346$ by Eq. (56), derived from the upper bound. It is thus seen that for low values of the impact velocity the elasto-plastic theory is needed to predict the behavior of the beam.

Fig. 7 gives the elasto-plastic moment diagram between t = 0 and $t = t_1 = 2.1$ for $v_0 = 67.4$. It is seen that in this case the high value of the initial velocity would create an elastic bending moment 10 times the value of the capacity moment.

Fig. 8 gives the plastic angle θ together with its upper and lower bounds, which for the high value 67.4 of the initial

velocity parameter are identical within the accuracy of the graphs except for the interval $t = 0$, $t_0 = 0.016$. For such small values of t_0 , the value of t_1 obtained from the bounds of θ coincide with the true value, as predicted by the theory.

It is thus seen that in the case of impulsive motion the approximate solutions based upon rigid-plastic behavior become acceptable whenever the maximum theoretical bending moment due to the impact is much larger, say 5 or more times larger, than the capacity moment, while elasto-plastic solutions are required in all other cases.

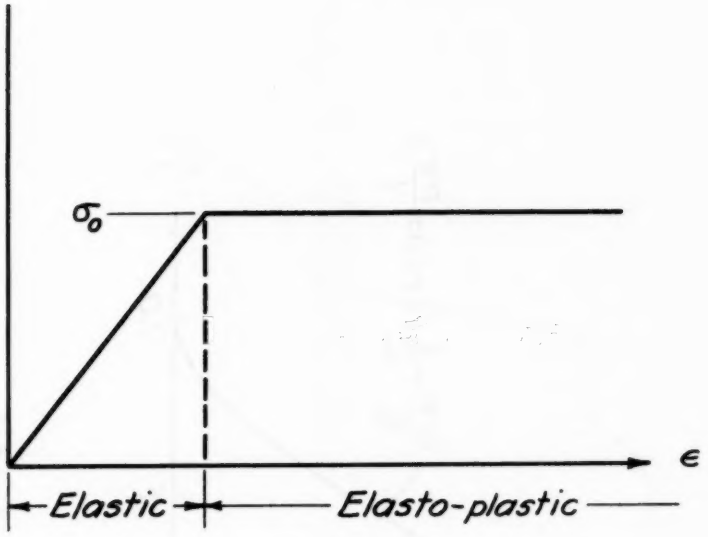


Fig. 1
ELASTIC AND ELASTO-PLASTIC
STATES OF STRESS

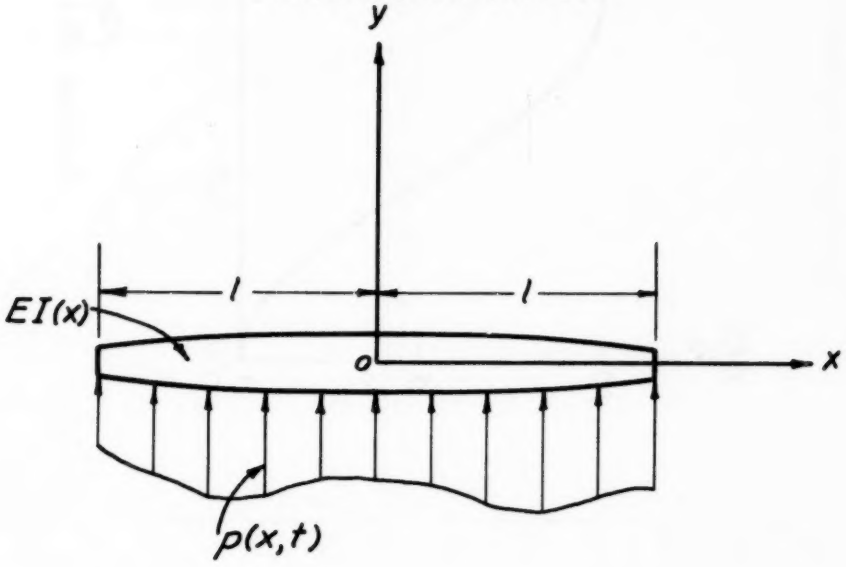


Fig. 2
FREE-FREE BEAM UNDER DYNAMIC FORCES

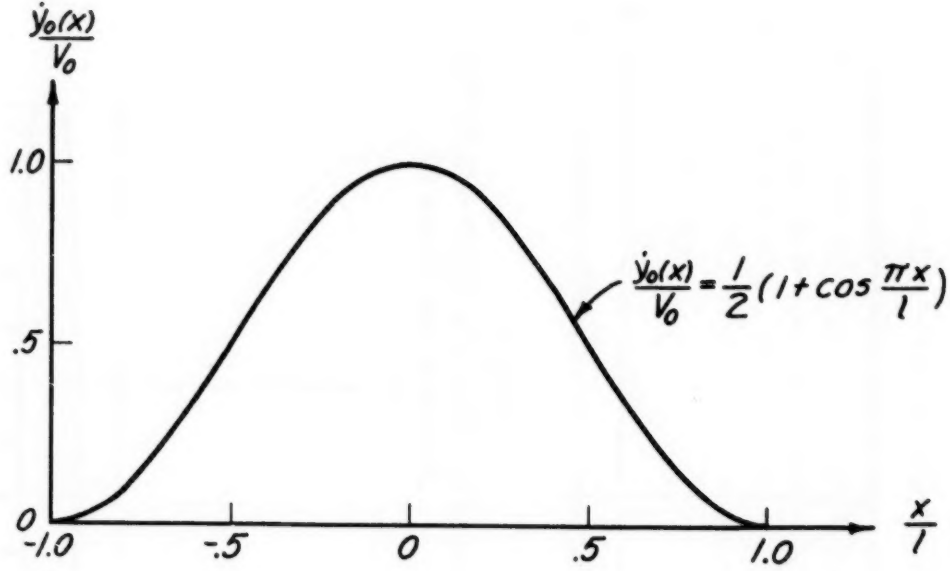
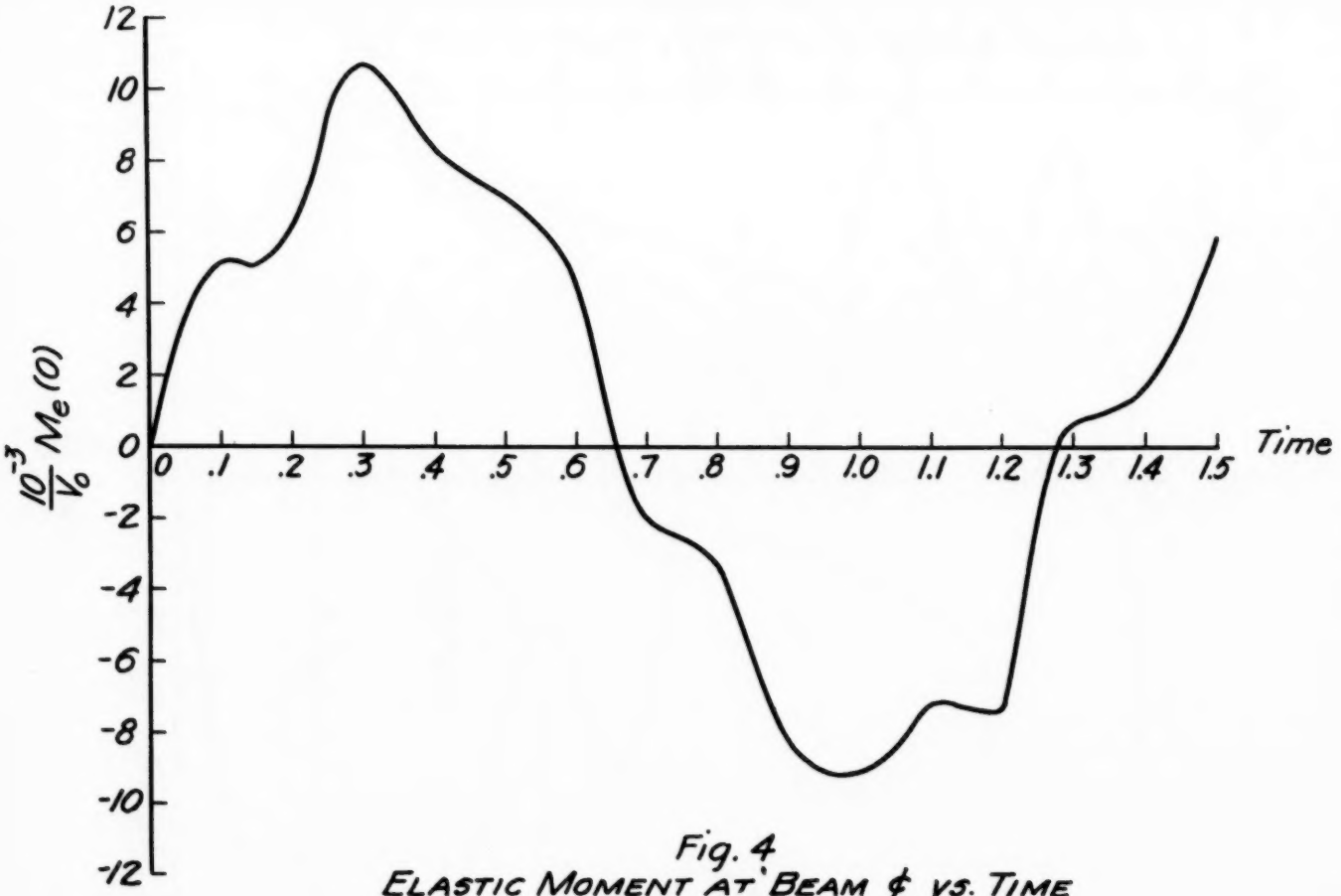


Fig.3
INITIAL VELOCITY DIAGRAM



287-21

Fig. 4
ELASTIC MOMENT AT BEAM ℄ VS. TIME

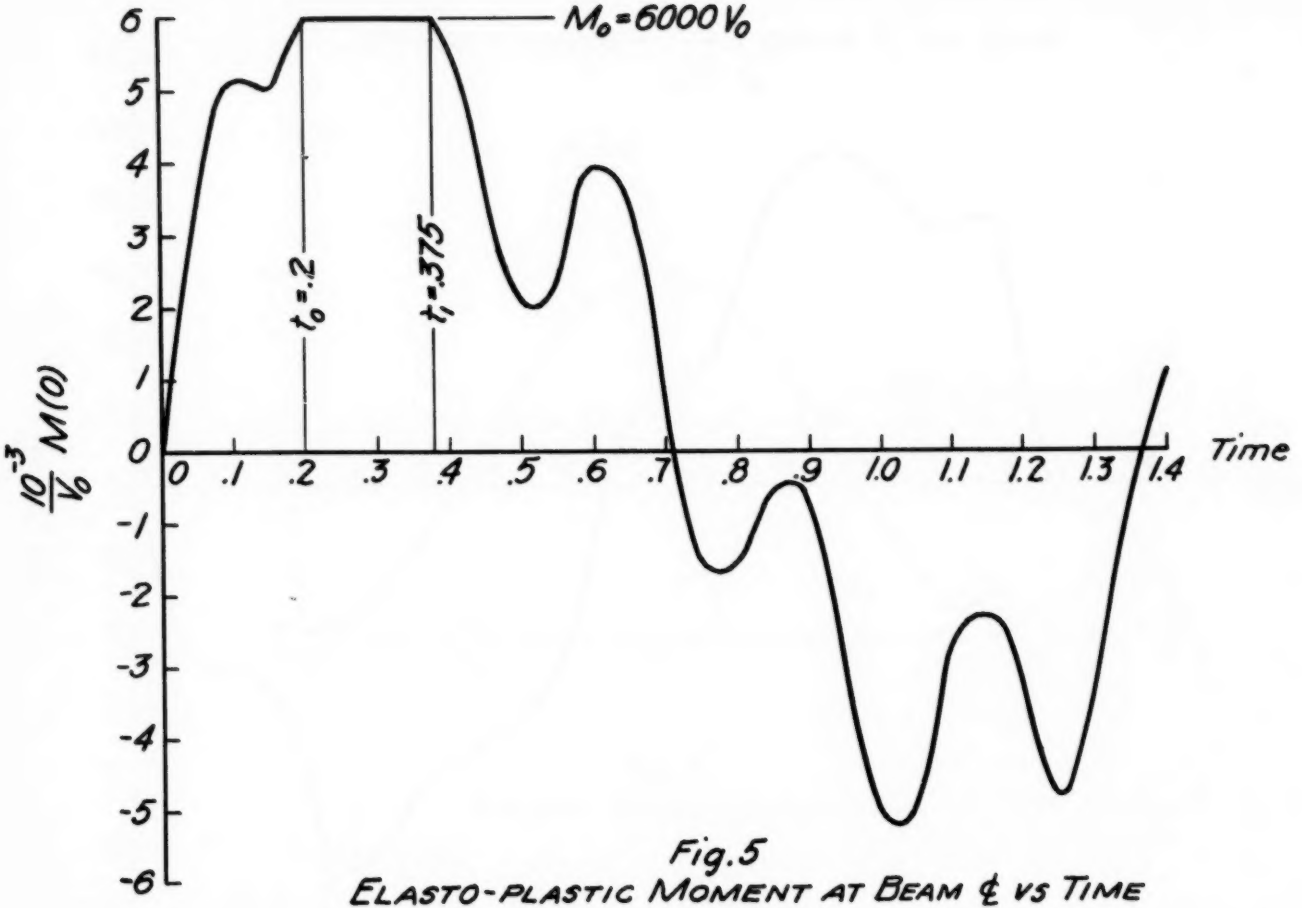


Fig. 5
ELASTO-PLASTIC MOMENT AT BEAM ℄ VS TIME
FOR $V_o = 11.1$

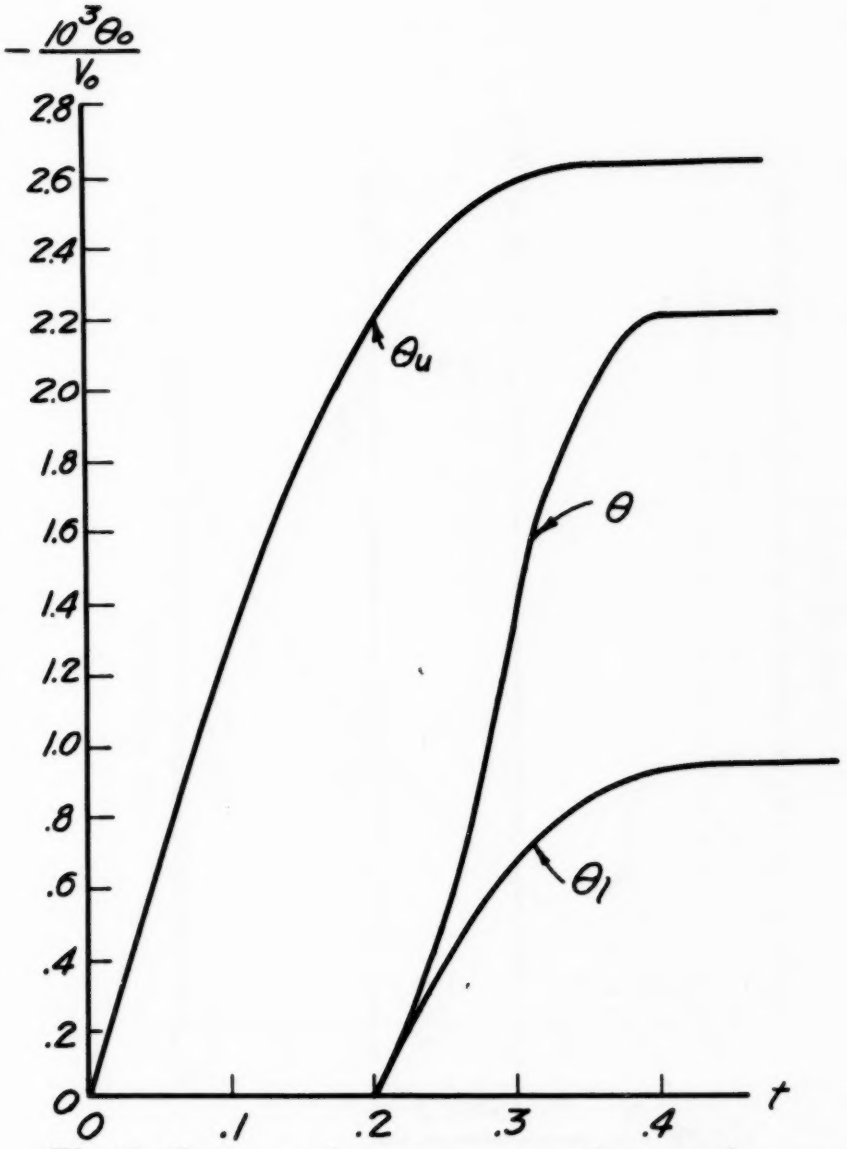


Fig. 6 Plastic Angle due to Cosine Impulse for $V_0 = 11.1$

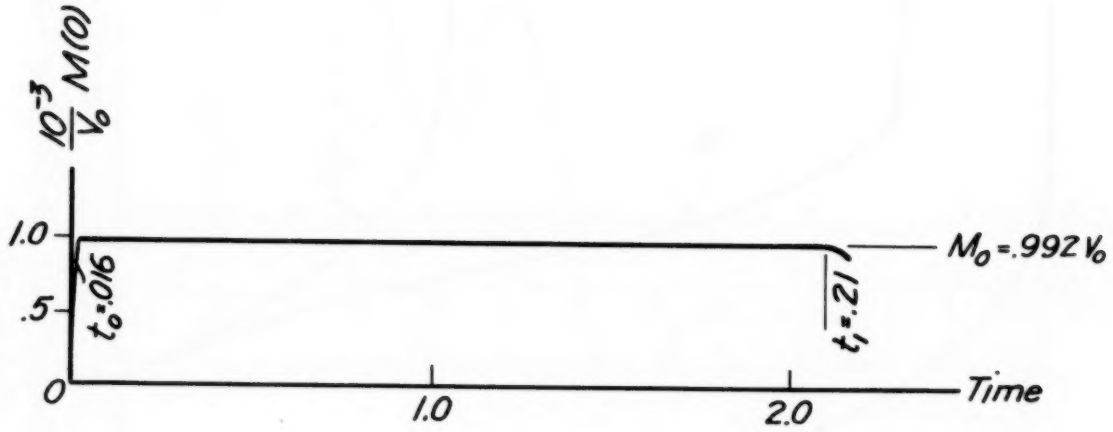


Fig. 7
ELASTO-PLASTIC MOMENT AT BEAM ℄ VS TIME
FOR V_0=67.4

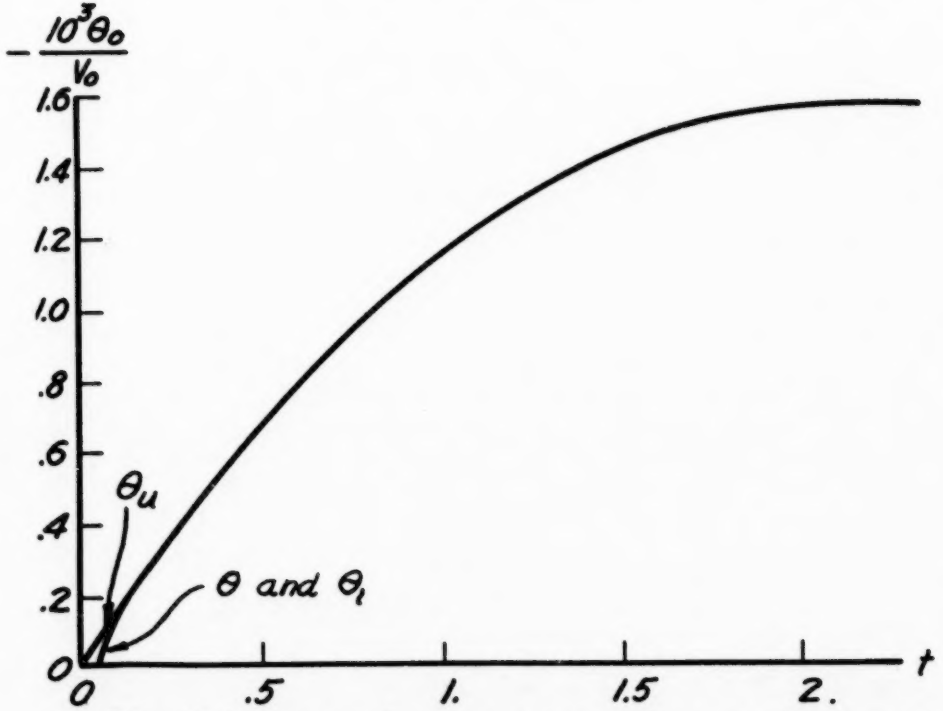


Fig. 8 PLASTIC ANGLE DUE TO COSINE IMPULSE FOR $V_o = 67.4$